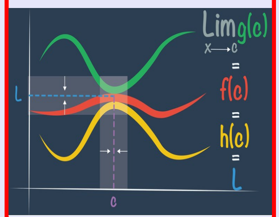
Calculus I

Lecture 10

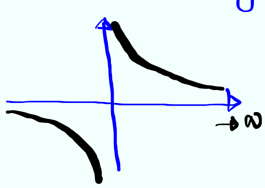


Feb 19-8:47 AM

Limits at $\pm\infty$

$\lim_{x\to\infty} f(x)$ or $\lim_{x\to -\infty} f(x)$

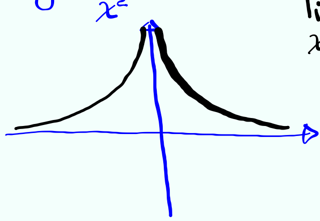
Take a look at $y=\frac{1}{x}$



$\lim_{x\to\infty} y = \lim_{x\to\infty}\frac{1}{x} = \boxed{0}$

$\lim_{x\to -\infty}\frac{1}{x} = \boxed{0}$

For $y=\frac{1}{x^2}$



$\lim_{x\to\infty}\frac{1}{x^2} = 0$

$\lim_{x\to -\infty}\frac{1}{x^2} = 0$

Mar 17-8:53 AM

$$\lim_{x\to\infty} \frac{x^2+4}{x^2+9} = \frac{\infty^2+4}{\infty^2+9} = \frac{\infty}{\infty} \quad \text{I.F.}$$

Divide by highest power of $x \Rightarrow x^2$

$$\lim_{x\to\infty} \frac{x^2+4}{x^2+9} = \lim_{x\to\infty} \frac{\frac{x^2+4}{x^2}}{\frac{x^2+9}{x^2}} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}+\frac{4}{x^2}}{\frac{x^2}{x^2}+\frac{9}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1+\cancel{\frac{4}{x^2}}\to 0}{1+\cancel{\frac{9}{x^2}}\to 0} = \frac{1}{1} = \boxed{1}$$

Evaluate $\lim_{x\to-\infty} \frac{x^2+3x-2}{2x^2-4x+5} = \frac{\infty}{\infty}$ I.F.

Divide by x^2

$$\lim_{x\to-\infty} \frac{x^2+3x-2}{2x^2-4x+5} = \lim_{x\to-\infty} \frac{\frac{x^2}{x^2}+\frac{3x}{x^2}-\frac{2}{x^2}}{\frac{2x^2}{x^2}-\frac{4x}{x^2}+\frac{5}{x^2}}$$

$$= \lim_{x\to-\infty} \frac{1+\cancel{\frac{3}{x}}\to 0-\cancel{\frac{2}{x^2}}\to 0}{2-\cancel{\frac{4}{x}}\to 0+\cancel{\frac{5}{x^2}}\to 0} = \boxed{\frac{1}{2}}$$

Mar 17-8:57 AM

Evaluate $\lim_{x\to\infty} \frac{10x-3}{\sqrt{25x^2-1}} = \frac{\infty}{\infty}$ I.F.

Divide by x

$$\lim_{x\to\infty} \frac{10x-3}{\sqrt{25x^2-1}} = \lim_{x\to\infty} \frac{\frac{10x-3}{x}}{\frac{\sqrt{25x^2-1}}{x}} = \lim_{x\to\infty} \frac{\frac{10x}{x}-\frac{3}{x}}{\sqrt{\frac{25x^2-1}{x^2}}}$$

As $x\to\infty$ $\quad x=\sqrt{x^2}$

$$= \lim_{x\to\infty} \frac{10-\frac{3}{x}}{\sqrt{\frac{25x^2}{x^2}-\frac{1}{x^2}}} = \lim_{x\to\infty} \frac{10-\cancel{\frac{3}{x}}\to 0}{\sqrt{25-\cancel{\frac{1}{x^2}}\to 0}} = \frac{10}{\sqrt{25}} = \frac{10}{5} = \boxed{2}$$

check:

choose $x=1000$

$$\frac{10x-3}{\sqrt{25x^2-1}} = \frac{10(1000)-3}{\sqrt{25(1000)^2-1}} \approx 1.99940004$$

If we choose $x=10000$

$$\frac{10(10000)-3}{\sqrt{25(10000)^2-1}} \approx 1.99994$$

$x\to\infty$

$\lim_{x\to\infty} \square \approx 2$

Mar 17-9:07 AM

Evaluate $\lim_{x\to-\infty} \frac{x+9}{\sqrt{9x^2-3x+8}} = \frac{-\infty}{\infty}$ I.F.

Divide by x

$$\lim_{x\to-\infty} \frac{x+9}{\sqrt{9x^2-3x+8}} = \lim_{x\to\infty} \frac{\frac{x}{x}+\frac{9}{x}}{\frac{\sqrt{9x^2-3x+8}}{x}}$$

$\boxed{\begin{array}{ll} \text{As } x\to\infty & x=\sqrt{x^2} \\ \text{As } x\to-\infty & x=-\sqrt{x^2} \end{array}}$

$$= \lim_{x\to-\infty} \frac{1+\frac{9}{x}}{\frac{\sqrt{9x^2-3x+8}}{-\sqrt{x^2}}} = \lim_{x\to\infty} \frac{1+\cancel{\frac{9}{x}}^{\to 0}}{-\sqrt{\frac{9x^2}{x^2}-\frac{3x}{x^2}+\frac{8}{x^2}}}$$

$9 \quad \cancel{\frac{3}{x}}^{\to 0} \quad \cancel{\frac{8}{x^2}}^{\to 0}$

$$= \frac{1}{-\sqrt{9}} = \boxed{-\frac{1}{3}}$$

Choose $x=-1000$: $\frac{-1000+9}{\sqrt{9(-1000)^2-3(-1000)+8}} \approx -.3302781448$

$x=-10000$: $\approx -.333\ldots$

$\approx \frac{-1}{3}$

Evaluate $\lim_{x\to\infty} 4x\cdot\sin\frac{2}{x}$ I.F.

$$= 4 \lim_{x\to\infty} \cancel{x}^{\to\infty} \cdot \cancel{\sin\frac{2}{x}}^{\to 0} = \infty\cdot 0$$

Choose $x=1000$: $4\cdot 1000\cdot\sin\frac{2}{1000} = 4000\cdot\sin\frac{1}{500}$

Radian Mode $\approx 7.99999\ldots$

≈ 8

$$4\lim_{x\to\infty} x\cdot\sin\frac{2}{x} = 4\lim_{x\to\infty}\frac{2\cdot\sin\frac{2}{x}}{2\cdot\frac{1}{x}}$$

$$= 4\cdot 2\lim_{x\to\infty}\frac{\sin\frac{2}{x}}{\frac{2}{x}} = 8\cdot\lim_{h\to 0}\cancel{\frac{\sin h}{h}}^{\to 1}$$

Let $h=\frac{2}{x}$

as $x\to\infty$, $h\to 0$

$$= 8\cdot 1 = \boxed{8}$$

I.F. $\Rightarrow \frac{0}{0}, \frac{\infty}{\infty}, \infty\cdot 0, \infty - \infty$

Evaluate $\lim_{x\to\infty}\left(\underbrace{\sqrt{x^2+4x}-x}\right) = \boxed{\infty-\infty}$

Conjugate $\sqrt{x^2+4x}+x$

let's try $x=10000$

$\sqrt{10000^2+4(10000)} - 10000 \approx 1.9998004 \approx 2$

$$\lim_{x\to\infty}\left[\sqrt{x^2+4x}-x\right] = \lim_{x\to\infty}\frac{\overset{A\ -B}{(\sqrt{x^2+4x}-x)}\overset{A\ +B}{(\sqrt{x^2+4x}+x)}}{\sqrt{x^2+4x}+x}$$

$$=\lim_{x\to\infty}\frac{(\sqrt{x^2+4x})^2-(x)^2}{\sqrt{x^2+4x}+x} = \lim_{x\to\infty}\frac{\cancel{x^2}+4x\cancel{-x^2}}{\sqrt{x^2+4x}+x}$$

$$=\lim_{x\to\infty}\frac{4x}{\sqrt{x^2+4x}+x} = \frac{\infty}{\infty}$$ Divide by x

$$=\lim_{x\to\infty}\frac{\frac{4x}{x}}{\sqrt{\frac{x^2}{x^2}+\frac{4x}{x^2}}+\frac{x}{x}} = \lim_{x\to\infty}\frac{4}{\sqrt{1+\cancel{\frac{4}{x}}^{\to 0}}+1}$$

$$=\frac{4}{\sqrt{1}+1} = \frac{4}{2} = \boxed{2}$$

Mar 17-9:38 AM

Evaluate $\lim_{x\to-\infty}\left(\sqrt{x^2-bx}+x\right) = \infty - \infty$ I.F.

Multiply & Divide by the Conjugate

$$\lim_{x\to-\infty}\frac{(\sqrt{x^2-bx}+x)(\sqrt{x^2-bx}-x)}{\sqrt{x^2-bx}-x}$$

$$=\lim_{x\to-\infty}\frac{x^2-bx-x^2}{\sqrt{x^2-bx}-x} = \lim_{x\to-\infty}\frac{-bx}{\sqrt{x^2-bx}-x} = \frac{\infty}{\infty}$$

Divide by x

As $x\to\infty$, $x=\sqrt{x^2}$

As $x\to-\infty$, $x=-\sqrt{x^2}$

$$=\lim_{x\to-\infty}\frac{\frac{-bx}{x}}{\frac{\sqrt{x^2-bx}}{x}-\frac{x}{x}}$$

$$=\lim_{x\to-\infty}\frac{-b}{-\sqrt{\frac{x^2}{x^2}-\frac{bx}{x^2}}-1} = \lim_{x\to-\infty}\frac{-b}{-\sqrt{1-\cancel{\frac{b}{x}}^{\to 0}}-1}$$

$$=\frac{-b}{-\sqrt{1}-1} = \frac{-b}{-2}$$

$$=\boxed{\frac{b}{2}}$$

Mar 17-9:49 AM

For the function $y = f(x)$

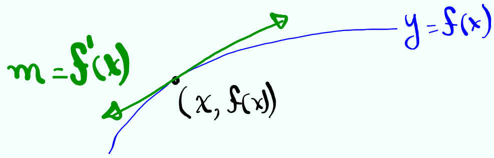
if $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ exists, that is called

$f'(x)$ F- prime of x

First Derivative of $f(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

It is the slope of the tangent line at any point on the graph of $y = f(x)$ if exists.



Mar 17-10:13 AM

Find $f'(x)$ of $f(x) = x^2 - 3x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)^2 - 3(x+h) - (x^2 - 3x)}{h}$$

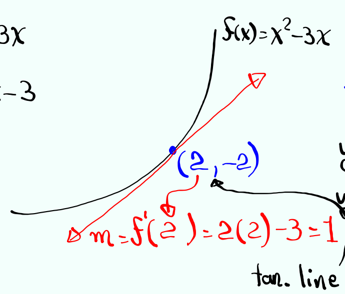
$$= \lim_{h \to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{3x} - 3h - \cancel{x^2} \cancel{+3x}}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}(2x + h - 3)}{\cancel{h}} = \lim_{h \to 0} (2x + h - 3)$$

$$= \boxed{2x - 3}$$

$f(x) = x^2 - 3x$

$f'(x) = 2x - 3$



$f(2) = 2^2 - 3(2) = -2$

$y - y_1 = m(x - x_1)$

$y - (-2) = 1(x - 2)$

$y + 2 = x - 2$

$\boxed{y = x - 4}$

Mar 17-10:18 AM

$f(x) = \sqrt{x}$, find $f'(x)$, $f(4)$, $f'(4)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} = \lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})(\sqrt{x+h} + \sqrt{x})}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{x} + \cancel{h}^{1} - \cancel{x}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

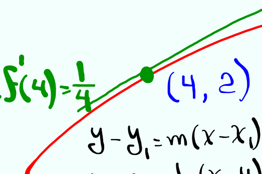
$$= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$f(x) = \sqrt{x}$ $\qquad f(4) = \sqrt{4} = 2$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\qquad f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}$

$f'(x)$ is defined for $x > 0$.

$f(x) = \sqrt{x}$



$m = f'(4) = \frac{1}{4}$ $\qquad (4, 2)$

$y - y_1 = m(x - x_1)$

$y - 2 = \frac{1}{4}(x - 4)$

$\to y = \frac{1}{4}x - 1 + 2$

$\boxed{y = \frac{1}{4}x + 1}$

tan. line at $(4,2)$

Mar 17-10:26 AM

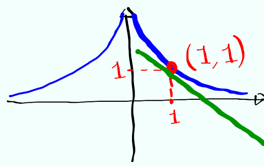
Other notation for first derivative

First Derivative
- y' $\quad$ y-Prime
- $\frac{dy}{dx}$ $\quad$ der. of y with respect to x
- $\frac{d}{dx}[f(x)]$

Mar 17-10:34 AM

Find equation of the tan. line to the graph of $f(x)=\frac{1}{x^2}$ at $x=1$.

$f(1)=\frac{1}{1^2}=1$



$y-1=-2(x-1)$

$\boxed{y=-2x+3}$

$m=f'(1)=\frac{-2}{1^3}=\boxed{-2}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{(x+h)^2}-\frac{1}{x^2}}{h}$$

$LCD=(x+h)^2\cdot x^2$

$$=\lim_{h\to 0}\frac{LCD\cdot\frac{1}{(x+h)^2}-LCD\cdot\frac{1}{x^2}}{h\cdot LCD}=\lim_{h\to 0}\frac{1\cdot x^2-(x+h)^2}{h(x+h)^2\cdot x^2}$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}-\cancel{x^2}-2xh-h^2}{h(x+h)^2x^2}=\lim_{h\to 0}\frac{\cancel{h}(-2x-h)}{\cancel{h}(x+h)^2\cdot x^2}$$

$$=\lim_{h\to 0}\frac{-2x-h}{(x+h)^2\cdot x^2}=\frac{-2x-0}{(x+0)^2\cdot x^2}=\frac{-2x}{x^2\cdot x^2}=\frac{-2x}{x^4}=\boxed{\frac{-2}{x^3}}$$

$f(x)=\frac{1}{x^2}$

$f'(x)=\frac{-2}{x^3}$

Mar 17-10:37 AM

Find all points on the graph of $f(x)=x^3+3x$ where we have horizontal tangent lines.

$m=0$

$\boxed{f'(x)=0 \text{ Solve}}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^3+3(x+h)-x^3-3x}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3\cancel{+3x}+3h\cancel{-x^3}\cancel{-3x}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(3x^2+3xh+h^2+3)}{\cancel{h}}$$

$$=\lim_{h\to 0}(3x^2+3xh+h^2+3)=3x^2+3$$

Solve $f'(x)=0$

$3x^2+3=0$

$3x^2=-3$

$x^2=\frac{-3}{3}$

$x^2=-1$

$x=\pm\sqrt{-1}$

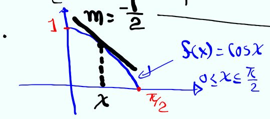
NO Real Solutions

$\boxed{\text{NO points with horizontal tan. lines.}}$

Mar 17-10:46 AM

Find the point on the graph of $f(x)=\cos x$ for $0 \le x \le \frac{\pi}{2}$ where slope of the tan. line is $-\frac{1}{2}$.

$m = \frac{-1}{2}$ → $\boxed{f'(x) = \frac{-1}{2}}$



$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{\cos x \cosh - \sin x \sinh - \cos x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\cos x \cosh - \cos x}{h} - \frac{\sin x \sinh}{h} \right]$$

$$= \lim_{h \to 0} \frac{\cos x (\cosh - 1)}{h} - \lim_{h \to 0} \frac{\sin x \sinh}{h}$$

$$= \cos x \underbrace{\lim_{h \to 0} \frac{\cosh - 1}{h}}_{0} - \sin x \underbrace{\lim_{h \to 0} \frac{\sinh}{h}}_{1}$$

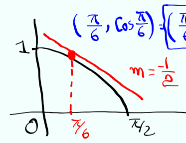
$$= -\sin x$$

$-\sin x = \frac{-1}{2}$

$\sin x = \frac{1}{2}$ $\quad x = 30^\circ$

$\boxed{x = \frac{\pi}{6}}$

$\left(\frac{\pi}{6}, \cos\frac{\pi}{6}\right) = \boxed{\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)}$



Mar 17-10:56 AM